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Monotone Approximation in Several Variables

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Let Ω denote the unit n-cube, $[0, 1]^n$, and let M be the set of all real valued functions on Ω which are nondecreasing in each variable. If f is a bounded Lebesgue measurable function on Ω and $1 < p < \infty$, let f_p denote the best L_p-approximation to f by elements of M. It is shown that f_p converges almost everywhere as p decreases to one to a best L_1-approximation to f by elements of M. If f is continuous, then f_p is continuous and converges uniformly as $p \downarrow 1$ to a best L_1-approximation to f by elements of M and f_p converges uniformly as $p \to \infty$ to a best L_∞-approximation to f by elements of M.

INTRODUCTION

For $n \geq 1$, let Ω be the unit n-cube, $[0, 1]^n$. Let μ denote n-dimensional Lebesgue measure on Ω, let Σ consist of the μ-measurable subsets of Ω and, for $1 \leq p \leq \infty$, let $L_p = L_p(\Omega, \Sigma, \mu)$. If $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ are elements of Ω, we write $x \leq y$ if $x_t \leq y_t$ for $1 \leq t \leq n$ and we write $x < y$ if $x_t < y_t$ for $1 \leq t \leq n$. A function $g: \Omega \to R$ is said to be nondecreasing in each variable if $x, y \in \Omega$ and $x \leq y$ imply that $g(x) \leq g(y)$. We will say that such a function is *nondecreasing*. Let M consist of all nondecreasing functions. For f in L_p and $1 \leq p \leq \infty$, let $\mu_p(f | M)$ denote the set of all best L_p-approximations to f by elements of M. Since M is a closed convex subset of the uniformly convex Banach space L_p, $1 < p < \infty$, $\mu_p(f | M)$ consists (up to equivalence) of exactly one function, which we denote by f_p. The function f is said to have the *Polya property* if $f_\infty = \lim_{p \to \infty} f_p$ is well defined as a bounded measurable function, i.e., if $p_n \to \infty$, then $\lim_{n \to \infty} f_{p_n}$ exists almost everywhere on Ω. If the above condition is true with ∞ replaced by 1, then f is said to have the *Polya-one* property. In Section 1, we show that, for any $n > 0$, and f in L_∞ has the Polya-one property. In Section 2, we assume that f is continuous and establish both the Polya and Polya-one properties, with uniform convergence in each case, and show that f_p is continuous, $1 \leq p \leq \infty$.

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1. The Polya-One Property

1.1 Theorem. *If $g \in M$, then g is continuous almost everywhere.*

Proof. Suppose l is a line in $\mathbb{R}^n$ parallel to the line in $\mathbb{R}^n$ joining $\bar{0} = (0,..., 0)$ and $\bar{1} = (1,..., 1)$ and $l \cap \Omega^0 \neq \emptyset$, where Ω^0 denotes the interior of Ω. Then there exist constants a_j, $j = 1,..., n$, and a such that $l \cap \Omega^0 = \{(t + a_1,..., t + a_n)\colon \ 0 < t < a\}$. Define h: $(0, a) \to \mathbb{R}$ by $h(t) = g(t + a_1,..., t + a_n)$. Suppose $0 < t_0 < a$, $x = (t_0 + a_1,..., t_0 + a_n)$ and g is discontinuous at x. Suppose without loss of generality that there exist $\varepsilon > 0$ and $\{x^i\} \subset \Omega \cap l$ with $x^i \downarrow x$ and, for each i, $g(x^i) > g(x) + \varepsilon$. Then, for any t in (t_0, a), there exists i such that $x^i = (x_1^i,..., x_n^i)$ satisfies

$$x_j < x_j^i < t + a_j, \qquad 1 \leqslant j \leqslant n$$

so

$$h(t) \geqslant g(x^i) > g(x) + \varepsilon = k(t_0) + \varepsilon$$

whence h is discontinuous at t_0. Since h is a nondecreasing function of one variable, there can be at most countably many points at which h is discontinuous. Thus, the one dimensional Lebesgue measure of the points of discontinuity of f on $l \cap \Omega$ is zero.

Let T: $\mathbb{R}^n \to \mathbb{R}^n$ be a linear isometry such that $T(\bar{1}) = (0, 0,..., 0, \sqrt{n})$. By Fubini's Theorem and the last paragraph the integral of the characteristic funtion of the image under T of the set of discontinuities of g is of n-dimensional Lebesgue measure zero. This concludes the proof of Theorem 1.1.

If $g \in M$, let $C(g)$ denote the set of all points of continuity of g. The following generalization of Helly's Theorem requires only minor modifications in the proof. (See [9, p. 221].)

1.2 Theorem. *If G is a uniformly bounded family of elements of M and K is an at most countable subset of Ω, then there exists a function g in M and a sequence $\{g_i\}$ in G such that $g_i(x) \to g(x)$ for every x in $C(g) \cup K$.*

Let $d_1(f, M) = \inf\{\|f - h\|_1\colon h \in M\}$, the distance from M to f.

1.3 Lemma. *M is an L_1-closed convex subset of L_1, and $\mu_1(f | M)$ is a nonempty subset of L_∞.*

Proof. Suppose $\{g_i\colon i = 1, 2,...\} \subset M$ and $g_i \to g$ in L_1. Since $\{g_i\}$ has a subsequence which converges to g almost everywhere, we may assume that $g_i \to g$ almost everywhere. Let $\bar{g} = \limsup_{i \to \infty} g_i$. Then $\bar{g} = g$ almost everywhere. Since each g_i is in M, $\bar{g}$ is in M. Thus g is equivalent to an element of M. Clearly M is convex.

Let $M' = \{h \in M: \|h\|_\infty \leq 2\|f\|_\infty\}$. Since $\inf\{\|f-h\|_1: h \in M'\} = d_1(f, M)$, there exists a sequence $\{g_i\} \subset M'$ such that $\|f-g_i\|_1 \to d_1(f, M)$. By Helly's Theorem $\{g_i\}$ has a subsequence which converges almost everywhere to a function g in M, so the Dominated Convergence Theorem shows that $g \in \mu_1(f|M)$. This concludes the proof of Lemma 1.3.

The next theorem shows that every bounded measurable function has the Polya-one property when M is the set from which best approximations are chosen. Let $f_1 = m_1(f|M)$, the unique element of $\mu_1(f|M)$ which minimizes

$$\left\{\int |f-h| \ln|f-h|: h \in \mu_1(f|M)\right\}.$$

The function f_1 is termed by Landers and Rogge the *natural* best L_1-approximation [8].

1.4 THEOREM. *If $f \in L_\infty$, then f_p converges almost everywhere as p decreases to one to an element of $\mu_1(f|M)$.*

Proof. We claim that $f_p \to f_1$ a.e. as $p \downarrow 1$. Suppose this is not the case. Then there exists a sequence $\{p_i\}$ such that $p_i \downarrow 1$ and there exists a subset E of Ω with $\mu E > 0$ and, for all x in E, $f_{p_i}(x)$ does not converge to $f_1(x)$.

Since $f_1 \in M$, Theorem 1.1 implies that there is a point y in Ω at which f_1 is continuous but $f_{p_i}(y)$ does not converge to $f_1(y)$. Thus, there exists a subsequence $\{q_i\}$ of $\{p_i\}$ such that

$$\lim_{i \to \infty} f_{q_i}(y) = d \neq f_1(y).$$

By [8, Theorem 2], f_{q_i} converges strongly in L_1 to f_1, so there is a subsequence $\{r_i\}$ of $\{q_i\}$ such that $f_{r_i} \to f_1$ almost everywhere. By Helly's Theorem there exist h in M and a subsequence $\{s_i\}$ of $\{r_i\}$ such that $f_{s_i} \to h$ on $C(h) \cup \{y\}$. Since $f_{s_i} \to f_1$ a.e., $f_1 = h$ a.e. Since $h(y) = d$ and f_1 is continuous at y and $h \in M$, either there exists an interval of the form $(y, z) = \{x \in \Omega: y < x < z\}$ such that x in (y, z) implies $h(x) > f_1(x)$ or there exists $w < y$ such that x in (w, y) implies $h(x) < f_1(x)$. In either case $\mu[f_1 \neq h] > 0$, a contradiction. This establishes Theorem 1.4.

There are two proper subsets N and P of M for which the result of Theorem 1.4 also holds. Characterizations of N and P require the following definition: If $g: \mathbb{R}^n \to \mathbb{R}$ and $a = (a_1, ..., a_n)$ and $b = (b_1, ..., b_n)$ are points in $\mathbb{R}^n$, let

$$\Delta_{b_i - a_i} g(a) = g(a_1, ..., a_{i-1}, b_i, a_{i+1}, ..., a_n) - g(a)$$

and let

$$\Delta^n_{b-a} g(a) = \Delta_{b_1 - a_1} \Delta_{b_2 - a_2} \cdots \Delta_{b_n - a_n} g(a).$$

A function $f\colon \Omega \to \mathbb{R}$ is said to be *positively monotone* if and only if, whenever $a, b \in \Omega$ and $b \geqslant a$,

$$\Delta^n_{b-a}\, g(a) \geqslant 0.$$

The set P will consist of all positively monotone functions in M. The set N consists of all positively monotone functions which vanish on the coordinate planes, i.e., $g(x_1,..., x_n) = 0$ if, for some i, $x_i = 0$. That $N \subset M$ is shown by Hildebrandt [4, p. 107]. That Helly's Theorem holds for N (with convergence everywhere) is shown in [1, Proposition 2–3]. N is closely related to the set of all distribution functions on $\mathbb{R}^n$.

Though the Polya-one property holds, the Polya property fails. The example in [2, Sect. 4] is easily modified to show this.

2. Uniform Polya Properties

In this section we restrict our attention to the case where $f \in C(\Omega)$, the set of real valued continuous functions on Ω. In this context, we show that both the Polya and Polya-one properties hold, with uniform convergence in each case, and that $f_p \in C(\Omega)$, $1 \leqslant p \leqslant \infty$.

We will reduce each question to a study of step functions. For convenience, we will use the dyadic rational partitions of Ω: for $k \geqslant 0$, let π_k denote the set of all points in Ω whose coordinates are rational numbers with denominator 2^k. The points of π_k divide Ω into a set of n-cubes $\{J(i)\colon i \in \mathscr{I}\}$, where each i has the form $i = (i_1, i_2,..., i_n)$ and each i_t is an integer in $[2^k, 2^k + k]$. We will henceforth call an n-cube a *cube*. The cubes $\{J(i)\colon i \in \mathscr{I}\}$ are pairwise disjoint and their union is Ω. If x and y are, respectively, the infimum and supremum of $J(i)$, then $J(i)$ contains the set

$$\{z\colon x_1 < z_1 \leqslant y_1, x_2 < z_2 \leqslant y_2,..., x_n < z_n \leqslant y_n\}$$

and any part of the boundary of $(x, y) = \{z\colon x < z < y\}$ which intesects the boundary of Ω. The point x will be called the *lower corner* of $J(i)$.

We will also denote by π_k the set of all such cubes. Let the cubes in π_k be partially ordered by restricting the order $\leqslant$ on Ω to the set of all centers of cubes in π_k. We assume that the order on π_k corresponds to the natural ordering on $\mathscr{I}$.

Let I_E denote the indicator function of a subset E of Ω, i.e., $I_E(x) = 1$ if $x \in E$ and $I_E(x) = 0$ otherwise, and let S_k consist of all functions $f\colon \Omega \to \mathbb{R}$ which have the form

$$f = \sum_{i \in \mathscr{I}} f(i)\, I_{J(i)}.$$

Let $\mathscr{L}^*$ consist of all subsets E of Ω which satisfy the condition $x \in E$, $x \leqslant y \Rightarrow y \in E$. Then $\mathscr{L}^*$ is a sigma lattice and M is the system of all functions measurable with respect to $\mathscr{L}^*$, i.e., $g \in M$ if and only if for each r in R, the set $[g > r]$ is an element of $\mathscr{L}^*$.

For each $J(i) \in \pi_k$, let $J^+(i) = \bigcup\{x \in \Omega\colon x \in J(j)$ and $J(j) \geqslant J(i)\}$. Let $\mathscr{L}_k$ be the sigma lattice generated by the set of intervals $\{J^+(i)\colon J(i) \in \pi_k\}$. Let M_k denote the set of all functions measurable with rspect to $\mathscr{L}_k$ and let $\mathscr{L}$ be the sigma lattice generated by $\bigcup_{k \geqslant 0} \mathscr{L}_k$.

2.1 Lemma. *Open sets in $\mathscr{L}^*$ are in $\mathscr{L}$.*

Proof. Suppose $C \neq \Omega$ is an open set in $\mathscr{L}^*$. For each $x \in C \cap \Omega^0$, choose a sequence $\{x^k\colon k \geqslant 0\}$ such that, for each k, x^k is the lower corner of a cube in π_k and $x^k \downarrow x$. For each $k \geqslant 0$, let $C_k = \bigcup\{x^k\colon x \in C\}$ and let $D_k = \bigcup I_x$, where the second union is over all x in C_k and $I_x = J^+(i)$, i being chosen so that x is the lower corner of $J(i)$. Then $D_k \in \mathscr{L}_k$ and $\bigcup\{D_k\colon k \geqslant 0\} = C$ so Lemma 2.1 holds.

Suppose $g \in M$. By Section 307 in [5], $\lim_{y \uparrow x} g(y)$ exists for every x in Ω^0. Define $g^*\colon \Omega \to \mathbb{R}$ by

$$
\begin{aligned}
g^*(x) &= \lim_{y \uparrow x} g(y), \qquad & \prod_{i=1}^{n} x_i \neq 0, \\
&= g(\bar{0}), & \prod_{i=1}^{n} x_i = 0.
\end{aligned}
$$

Then for each $r \in R$, $[g^* > r]$ is an open element of $\mathscr{L}^*$, so $[g^* > r]$ is in $\mathscr{L}$. By Theorem 1.1, g is continuous almost everywhere, so g is equivalent to an $\mathscr{L}$-measurable function. For $1 < p < \infty$, L_p is a uniformly convex Banach space so, for each f in $C(\Omega)$ there exist unique nearest points f_p^k in M_k, f_p in M and $\hat{f}_p$ in the set of $\mathscr{L}$-measurable functions. The unicity is, of course, up to equivalence. The above shows that $\hat{f}_p = f_p$ almost everywhere so we may assume without loss of generality that f_p is $\mathscr{L}$-measurable.

2.2 Lemma. *If $1 < p < \infty$ and $f \in S_k$, then f_p^{k+1} is (up to equivalence) also in S_k.*

Proof. We will assume that the statement is false and derive a contradiction by constructing an element of $\mu_p(f \mid M_{k+1})$ which is not equivalent to f_p^{k-1}. Let $\mathscr{I}$ and $\mathscr{I}'$ be the index sets such that $\pi_k = \{J(i)\colon i \in \mathscr{I}\}$ and $\pi_{k+1} = \{J(i)\colon i \in \mathscr{I}'\}$. Let $S = \{|f_p^{k+1}(i) - f_p^{k+1}(j)|\colon i, j \in \mathscr{I}'\}$ and $T = \{|f_p^{k+1}(i) - f(i)|\colon i \in \mathscr{I}'\}$. Let σ (respectively, τ) be the smallest positive number in S (respectively, T). We may assume without loss of generality that $\min\{\sigma, \tau\} = 2$. For any v in $\mathscr{I}'$, let $v' = (v_1 + 1, v_2, \ldots, v_n) \in \mathscr{I}'$.

If there exists β in $\mathscr{I}$ such that f_p^{k+1} is not constant on $J(\beta)$, then there exists α in $\mathscr{I}'$ such that $J(\alpha) \cup J(\alpha') \subset J(\beta)$ and (relabeling if necessary) $f_p^{k+1}(\alpha') > f_p^{k+1}(\alpha)$. We now construct a pair of sets on which we will alter the value of f_p^{k+1}. Let

$$A = \{j \in \mathscr{I}': j_1 = \alpha_1, j_2 \geqslant \alpha_2, ..., j_n \geqslant \alpha_n \text{ and } f_p^{k+1}(j) = f_p^{k+1}(\alpha)\},$$

$$B = \{j \in \mathscr{I}': j_1 = \alpha_1 + 1, j_2 \leqslant \alpha_2, ..., j_n \leqslant \alpha_n \text{ and } f_p^{k+1}(j) = f_p^{k+1}(\alpha')\},$$

$$A' = A \cup \{(j_1 - 1, j_2, ..., j_n): j \in B\} \subset \mathscr{I}',$$

$$B' = B \cup \{(j_1 + 1, j_2, ..., j_n): j \in A\} \subset \mathscr{I}',$$

$$A^* = \bigcup \{J(j): j \in A'\},$$

$$B^* = \bigcup \{J(j'): j \in B'\}.$$

Then A' and B' have the same number of elements and for each $j \in A'$, $f(j) = f(j')$, so A^* may be written as a disjoint union of sets A_1^*, A_2^* and A_3^*, where, for each cube $J(j) \subset A_1^*$, $f_p^{k+1}(j) < f(j) < f_p^{k+1}(j')$, for each $J(j) \subset A_2^*$, $f(j) < f_p^{k+1}(j) < f_p^{k+1}(j')$ and for each $J(j) \subset A_3^*$, $f_p^{k+1}(j) < f_p^{k+1}(j') < f(j)$. For $r = 1, 2, 3$, let $B_r^* = \bigcup \{J(j'): J(j) \in A_r^*\}$.

We now construct an element of M_{k+1} which is a better L_p-approximation to f than is f_p^{k+1}: *Define* $\psi: \Omega \to \mathbb{R}$ by

$$\begin{aligned} \psi(x) &= f_p^{k+1}(x) \quad , \qquad x \notin A^*, \\ &= f_p^{k+1}(x) + 1, \qquad x \in A^*. \end{aligned}$$

Clearly $\psi \in M_{k+1}$. If $\|f - \psi\|_p \leqslant \|f - f_p^{k+1}\|_p$, then f_p^{k+1} is not the unique best L_p-approximation to f by elements of M_{k+1}, a contradiction. If $\|f - \psi\|_p > \|f - f_p^{k+1}\|_p$, define $\psi': \Omega \to \mathbb{R}$ by

$$\begin{aligned} \psi'(x) &= f_p^{k+1}(x) \quad , \qquad x \notin B^*, \\ &= f_p^{k+1}(x) - 1, \qquad x \in B^*. \end{aligned}$$

Then $\psi' \in M_{k+1}$ and we claim that $\|f - \psi'\|_p < \|f - f_p^{k+1}\|_p$. Indeed, let

$$e_r = \int_{A_r^*} |f - f_p^{k+1}|^p - \int_{A_r^*} |f - \psi|^p$$

and

$$e'_r = \int_{B_r^*} |f - f_p^{k+1}|^p - \int_{B_r^*} |f - \psi'|^p$$

for $r = 1, 2, 3$. By the construction of the A_r^*, $e_1 \geqslant 0$, $e'_1 \geqslant 0$, $e_2 < 0$,

$e'_2 > -e_2$, $e_3 > 0$ and $e'_3 > -e_3$. Since $\|f-\psi\|_p > \|f-f_p^{k+1}\|_p$, $e_1+e_2+e_3 < 0$. Thus

$$\int_{B^*} |f-f_p^{k+1}|^p - \int_{B^*} |f-\psi'|^p = e'_1+e'_2+e'_3 > e'_1+e_1 \geqslant 0,$$

so $\|f-\psi'\|_p \leqslant \|f-f_p^{k+1}\|_p$. Therefore, in every possible case, the assumption that f_p^{k+1} is not essentially constant on $J(\alpha)$ produces a contradiction. Hence, Lemma 2.2.

2.3 THEOREM. *If $f \in S_k$, then $f_p \in S_k$ for all p, $1 < p < \infty$.*

Proof. Our first claim is that for each integer $m > k$, $f_p^m \in S_k$. Indeed, suppose $m > k$ and there exists $J(\alpha)$ in π_k such that f_p^m is not constant on $J(\alpha)$. Now a construction similar to that in the proof of Lemma 2.2 produces a contradiction.

By the construction of $\mathscr{L}$, the sequence $\{\mathscr{L}_k\}$ of sigma lattices increases to $\mathscr{L}$. Thus, by [7, Theorem 4.1], the constant sequence $\{f_p^m : m > k\}$ converges almost everywhere to f_p. This proves Theorem 2.3.

Theorem 2.3 effectively allows us to restrict our attention to a function whose domain is a finite partially ordered set. For such a function several properties are known: see [3, 6]. The proof in those papers are easily adapted to yield the following theorem.

2.4 THEOREM. *Let $f \in C(\Omega)$. Then there exists nondecreasing functions f_p, $1 \leqslant p \leqslant \infty$, such that, for $1 < p < \infty$, f_p is (up to equivalence) the best L_p-approximation to f by nondecreasing functions,*

$$\lim_{p \downarrow 1} f_p = f_1$$

and

$$\lim_{p \to \infty} f_p = f_\infty,$$

with uniform convergence in each case.

2.5 THEOREM. *The nondecreasing functions f_1 and f_∞ are elements of $\mu_1(f|M)$ and $\mu_\infty(f|M)$, respectively.*

Proof. That $f_1 \in \mu_1(f|M)$ follows from Theorems 1.4 and 2.4. For f_∞, suppose $g \in M$ satisfies $\|f-g\|_\infty < \|f-f_\infty\|_\infty$. Then there exist real numbers a and b such that $\|f-g\|_\infty < a < b < \|f-f_\infty\|_\infty$ so, for sufficiently large p, $\|f-g\|_p < a$. By Theorem 2.4, $f-f_p \to f-f_\infty$ uniformly, so there exists a set E with $\mu E > 0$ such that, for sufficiently large p, $|f(x) - f_p(x)| > b$ for every x in E. Thus, for large p,

$$\|f - f_p\|_p = \left\{ \int |f - f_p|^p d\mu \right\}^{1/p} \geqslant b(\mu E)^{1/p} > a,$$

a contradiction. This concludes the proof of Theorem 2.5.

We now turn to the question of the continuity of f_p, $1 \leqslant p \leqslant \infty$. Our approach is to uniformly approximate f by functions in S_k, $k \geqslant 1$. The following definitions will expedite our discussion of these step functions. We will say that $J(i)$ and $J(j)$ in π_k are *adjacent* if $j = (i_1, i_2, ..., i_{t-1}, i_t \pm 1, i_{t+1}, ..., i_n)$ for some t, $1 \leqslant t \leqslant n$. The union of a set of cubes is said to be a *component* if for any two cubes $J(i)$ and $J(j)$ in the set, there exist cubes $J(i^1) = J(i)$, $J(i^2), ..., J(i^m) = J(j)$ such that, for $1 \leqslant t \leqslant m-1$, $J(i^t)$ is adjacent to $J(i^{t+1})$. If $J(i)$ and $J(j)$ are any two adjacent cubes, we will call $|g(i) - g(j)|$ a *jump* of g.

2.6 LEMMA. *For any $\varepsilon > 0$, if $f \in S_k$ and f has no jump greater than ε, then, for $1 < p < \infty$, f_p has no jump greater than 3ε.*

Proof. By Lemma 2.2, $f_p = f_p^k$. If there exist adjacent cubes $J(\alpha)$ and $J(\alpha')$ in π_k such that $f_p(\alpha') - f_p(\alpha) > 3\varepsilon$, then an element of $\mu_p(f|M) = \mu_p(f|M_k)$ which is not equivalent to f_p can be constructed in a manner similar to the construction in Lemma 2.2. In the amended proof, the role of $J(\beta)$ is played by the cube in π_{k-1} which contains $J(\alpha)$, each occurrence of "$k+1$" is replaced by "k" and, for each cube $J(j) \subset A_1^*$ (respectively, A_2^*, A_3^*), $f_p(j) < f(j) < f(j') < f_p(j')$ (respectively, $f(j) \leqslant f_p(j)$, $f(j') \geqslant f_p(j')$).

2.7 THEOREM. *If f: $\Omega \to R$ is continuous and $1 \leqslant p \leqslant \infty$, then f_p is continuous.*

Proof. In view of Theorem 2.4, it suffices to prove the statement for $1 < p < \infty$. Let $\varepsilon > 0$ be given. Then there exist $k = k(\varepsilon) > 0$ such that $\sup_{x \in J} f(x) - \inf_{x \in J} f(x) < \varepsilon$ for every J in π_k. Define f^ε by

$$f^\varepsilon(x) = \sup_{x \in J} f(x), \qquad x \in J \in \pi_k,$$

and define f_ε similarly, with "sup" replaced by "inf." Since $f_\varepsilon \leqslant f \leqslant f^\varepsilon$ and $f^\varepsilon - \varepsilon \leqslant f \leqslant f_\varepsilon + \varepsilon$, the monotonicity of the nearest point projection (see 2.8 in [7]) implies that

$$(f_\varepsilon)_p \leqslant f_p \leqslant (f^\varepsilon)_p,$$
$$(f^\varepsilon)_p - \varepsilon \leqslant f_p \leqslant (f_\varepsilon)_p + \varepsilon$$

and

$$(f^{\varepsilon})_p - (f_{\varepsilon})_p \leqslant 2\varepsilon.$$

Since neither f_{ε} nor f^{ε} has a jump greater than ε, Lemma 2.6 implies that neither $(f_{\varepsilon})_p$ nor $(f^{\varepsilon})_p$ has a jump greater than 3ε.

Let B be a ball in Ω of radius 2^{-k-1}. Then

$$\sup_{x \in B} f_p(x) - \inf_{x \in B} f_p(x) \leqslant 5n\varepsilon,$$

whence f_p is continuous.

References

1. R. B. Darst and Z. Honargohar, Weak-star convergence in the dual of the continuous functions of the n-cube, $1 \leqslant n \leqslant \infty$, *Trans. Amer. Math. Soc.* **275** (1983), 357–372.
2. R. B. Darst and R. Huotari, Best L_1-approximation of bounded, approximately continuous functions on [0, 1] by nondecreasing functions, *J. Approx. Theory* **43** (1985), 178–189.
3. R. B. Darst and S. Sahab, Approximation of continuous and quasicontinuous functions by monotone functions, *J. Approx. Theory* **38** (1983), 9–27.
4. T. H. Hildebrandt, "Introduction to the Theory of Integration," New York, 1963.
5. Hobson, E. W., "The Theory of Functions of a Real Variable," New York, 1957.
6. R. Huotari, Best L_1-approximation of quasi-continuous functions on [0, 1] by nondecreasing functions, *J. Approx. Theory* **44** (1985), 221–229.
7. D. Landers and L. Rogge, Isotonic approximation in L_s, *J. Approx. Theory* **31** (1981), 199–223.
8. D. Landers and L. Rogge, Natural Choice of L_1-approximants, *J. Approx. Theory* **33** (1981), 268–280.
9. I. P. Natanson, "Theory of Functions of a Real Variable, Vol. 1," Ungar, New York, 1955.